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Fractal Dimension at the Phase Transition of Inhomogeneous Cellular Automata

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For random binary mixtures of cellular automata in the square lattice, calculations are made of the fractal dimensions associated with the damage spreading and the propagation time of damage at the transition to chaos. Two rules are mixed and universalities of these quantities are sought with respect to change of the rules.

KEY WORDS: Cellular automata; fractal dimensions; universality.

1. INTRODUCTION

Inhomogeneous cellular automata have been the subject of much interest in the recent years.[(1)] In this model one associates to each site of a given lattice a Boolean variable σ_i which takes the values 0 or 1. The evolution for each site σ_i is determined by a rule randomly chosen among the permitted Boolean functions of K inputs, these being the lattice neighbor sites of σ_i. This model presents a transition between a chaotic and frozen phase.[(2,3)] The frozen phase is characterized by the fact that it is stable with respect to perturbations (damages), whereas in the chaotic phase it is not.

At the transition point some quantities are fractal.[(2,4,5)] In particular, de Arcangelis and Stauffer[(4)] showed numerically that triangular and square lattices of the Kauffman model have the same critical exponents and they believe that this property is universal in relation to other lattices. Da Silva *et al.*[(3)] checked for many mixtures of only two rules whether a transition to chaos occurs. Now I find the fractal dimensions at this transition and instead of changing the lattices I vary the rules.

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2. METHOD

I calculate the phase transition in two ways, using the damage[6] as a disorder parameter. This quantity is defined as the global effect observed in the time evolution of the system due the introduction of a single error. Quantitatively one can measure the damage by counting the number of sites differing between two samples that evolve simultaneously subjected to the same rules and which are different in only one or few spins at time $t=0$.

In the first method I use the actual damage[6] and as initial damage I take only one randomly selected spin. The value of p_c for initial damage going to zero is obtained by the use of the extrapolation function in ref. 6. In the second case I calculate p_c by the use of the "damage spreading in a gradient" method.[7] In this method the choice of the initial damage is more arbitrary. I use as initial damage one entire line because the convergence is faster, and as disorder parameter the total damage.[6] The values of p_c obtained by the two methods are compatible.

I am interested in calculating two fractal dimensions defined at $p=p_c$ the

$$M \sim L^D$$

and

$$\tau \sim L^{D'}$$

where M is the actual damage at $t=\tau$, τ being the time for a damage starting in the center to propagate to the boundaries, and L is the size of the system.

For the $L \times L$ square lattice I check how long it takes for the damage to reach the top of the system. This time I call τ and the corresponding actual damage M.

Once I have p_c, I calculate M and τ taking as initial damage a whole lattice line in the center of the square lattice because then statistics is better.[5]

3. RESULTS

I start by analyzing the random mixture between the five-input generalized OR function [which is true (1) if at least one of its five arguments is true] and the five-input generalized XOR function[3] (which is true if an odd number of its arguments is true). These five sites are the central site and its neighbors. I mix these two rules randomly in such way that one has OR with probability p and XOR with probability $1-p$. The phase transition frozen–chaotic occurs for $p_c = 0.390 \pm 0.002$.

Figure 1 shows the L evolution for the effective exponents $D(L)$, $D'(L)$ with L the size of the system. I obtain the limit $(L \to \infty)$ values $D(\infty) \approx 1.882$ and $D'(\infty) \approx 1.096$.

Similarly, I studied three more mixtures of two binary variable functions. In the first I take with probability p the rule that gives true if an even number of neighbor sites is true and false (0) otherwise and with probability $1-p$ the rule which is true only when three or four of the four input sites are true. The onset to chaos occurs at $p_c = 0.465 \pm 0.002$. The results corresponding to Fig. 1 are shown in Fig. 2 with $D(L) = 1.905$ and $D'(L) = 1.163$.

As the third example I study the rules used by Hartman and Vichniac[8] to discuss inhomogeneous automata. In this case I take with probability p the five-input XOR rule and with probability $1-p$ the four-input logical function AND. The onset to chaos is obtained for

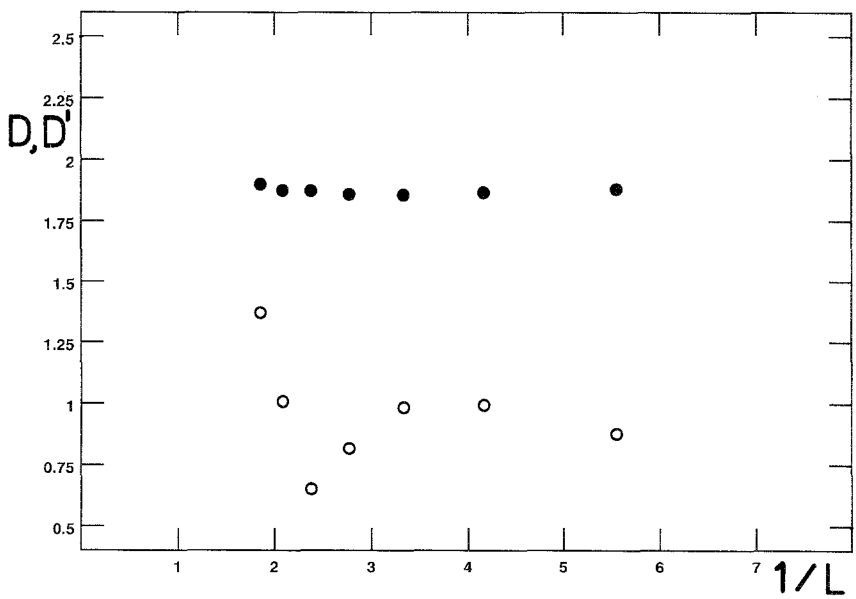

Fig. 1. Fractal dimensions D (●) and D' (○) at the phase transition in which five-input rules XOR (concentration p) and OR (concentration $1-p$) are randomly mixed, as function of $1/L$. The transition to chaos occurs at $p_c = 0.390 \pm 0.002$. The data are with error bars $\Delta = 0.02$ for D and $\Delta' = 0.1$ for D' for small L ($L \approx 300$), increasing with the size L. I used 10,000 samples for $L = 180$, decreasing to 1000 samples for $L = 600$. Straight-line fits of these points give $D = 1.882$ and $D' = 1.096$ for $L \to \infty$.

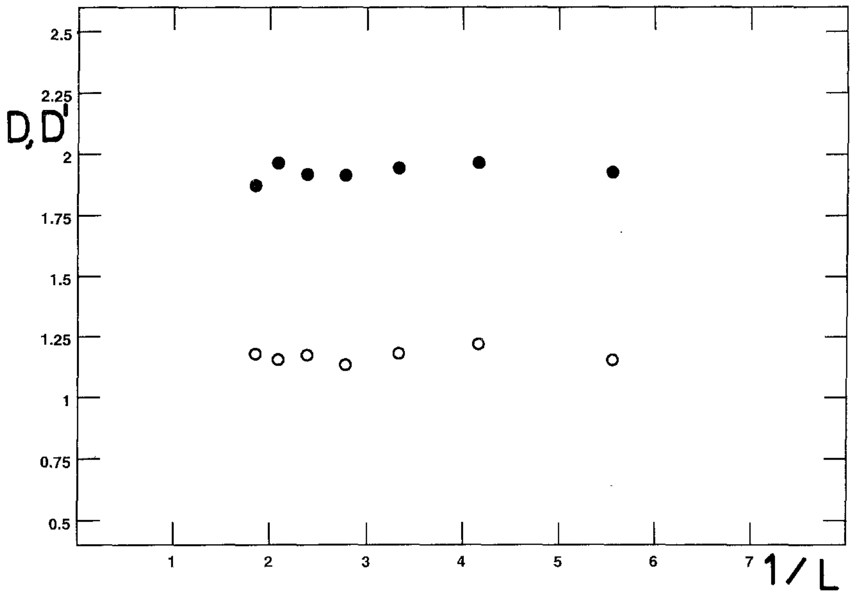

Fig. 2. Result of mixing the four-input functions NOT(XOR) (concentration p) and the rule that is true only when three or four sites are true (concentration $1-p$). One gets $p_c = 0.465 \pm 0.002$. Statistics and error bars are the same as in Fig. 1. Here $D = 1.905$ and $D' = 1.163$ for $L \to \infty$.

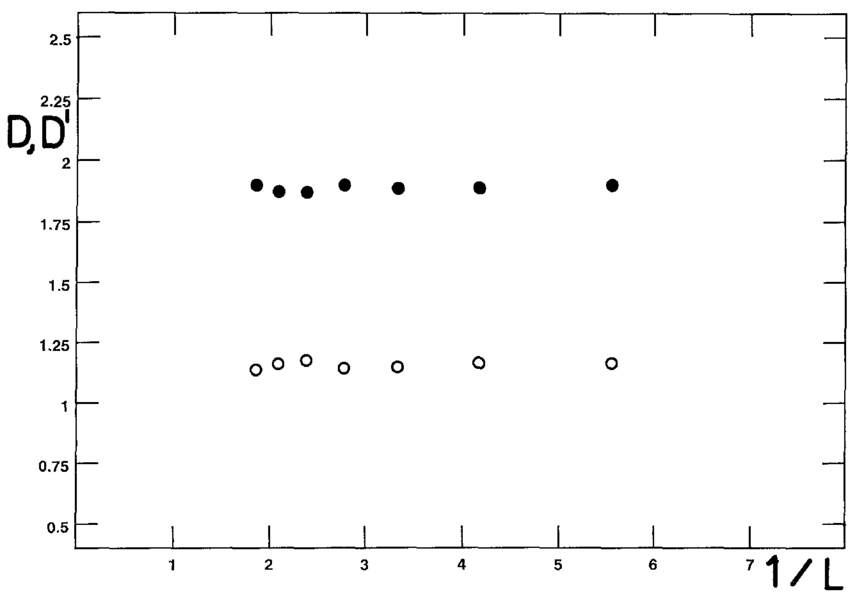

Fig. 3. Analogous results to Figs. 1 and 2. The four-input XOR (probability p) is mixed with the four-input function AND (probability $1-p$). One obtains $p_c = 0.577 \pm 0.002$. Statistics and error bars are the same as in the previous examples. $D = 1.878$ and $D' = 1.148$ for $L \to \infty$.

$p_c = 0.577 \pm 0.002$. The values for $D(L) = 1.878$ and $D'(L) = 1.148$ can be seen in Fig. 3.

I have also analyzed the rules used by Stauffer[9] to discuss the concept of "forcing rules." One takes with probability p the rule that is true if an even number of neighbor sites is true and with probability $1-p$ the rule that is true if at most two of the four neighbors are true. Analogous results as in the previous examples are found at $p_c \approx 0.33$.

Thus, the thresholds for the transition to chaos disagree with the random percolation threshold $p_c = 0.593$ or $p_c = 0.407$. However, the exponents D and D' agree with the well-known fractal dimensions 1.896 and 1.1 of the mass or the chemical distance in random percolation. For the Kauffman model, these exponents are 1.6 and 1.5. In other words, binary mixtures of cellular automata are in the universality class of random percolation, and not in the universality class of the Kauffman model.

4. CONCLUSIONS

I have studied the critical behavior at the transition to chaos of several binary mixtures of cellular automata. I analyzed the fractal dimensions associated with the damage spreading and the propagation time of damage, D and D', respectively. For the four cases studied I found these quantities to be universal. I showed that the universality class of these automata is the same as the universality class of random percolation and not that of the Kauffman model as one have might guessed. I believe that this result is general with respect to other combinations of binary mixtures of cellular automata.

The study of other combinations of rules or a general proof of the results presented here would be interesting.

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